A Note on the Behavior of Long Zero Coupon Rates in a No Arbitrage Framework

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ABSTRACT. This paper investigates the behavior of long zero-coupon rates and its consequences for usual arbitrage models of the term structure. Following Dybvig, Ingersoll and Ross (1996), we show that the long-term rate (i.e. the yield of a zero-coupon bond of infinite maturity) is either constant (as assumed in many existing models) or is necessarily a non-decreasing process. In the light of this discussion, we first review some particular models including the Affine Yield, Linear Gaussian and Quadratic Gaussian classes. We show how these models should be corrected to take into account the behavior of the long zero-coupon rate. Some properties of these new models are rather inconsistent with practical intuition, which leaves the issue somewhat unresolved. Secondly we describe the operational implications of this issue and show that it may shed light on the calibration procedure of term structure models.

Key words: Term structure of interest rates; Long term interest rates; Consol rates; Factor models of interest rates.

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1. INTRODUCTION

This paper analyzes the consequences of the behavior of the long zero-coupon rate for the usual models of the term structure. Indeed Dybvig, Ingersoll and Ross (1996) show that the long zero-coupon rate is necessarily a non-decreasing process. Their proof is provided in full generality, without specifying a given model of interest rates. Surprisingly, this remarkable result has not captured the attention it deserves, both among academics and practitioners. It is this dual viewpoint that has also motivated for some time now our research on the properties of the term structure dynamics (see El Karoui and Geman (1991), (1994)) mostly in a state variable approach (see El Karoui and Lacoste (1992), Frachot and Lesne (1993)). The goal of this paper is to investigate, in this state variable approach, the result of Dybvig, Ingersoll and Ross (1996).

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Indeed, since the paper by Brennan and Schwartz (1979) - which raises additional issues we do not want to discuss in this paper - the long term rate (i.e., the consol rate) has been a natural candidate for being a state variable in a multifactor model. It is indeed in this framework of factor models that we want to investigate the intriguing properties of the long zero-coupon rate; moreover, we try to derive from them empirical and econometric implications for the yield curve. The long zero coupon rate is defined as the limit of the yield of a zero-coupon bond when its maturity goes to infinity. In particular, we are not focusing on the consol rate, i.e., the yield of a perpetual annuity, even though we will allude to it when its consideration becomes relevant to our problem.

By "usual models", we mean the so-called factor models of the term structure including those of Vasicek (1977), Cox, Ingersoll, and Ross (1985), Jamshidian (1989), Duffie and Kan (1996). For sake of simplicity, we shall nest all these models into two specific classes that we will refer to as the "Affine Yield" and "Quadratic Gaussian" classes. Both are factor models of the term structure i.e., yields of all maturities are represented as functions of a small number of state variables whose evolution follows a Markov diffusion process. They differ from the functional link between yields and state variables; in the Affine Yield class, this link is linear whereas it is quadratic in the Quadratic Gaussian class. The Affine Yield models have been first defined by Duffie and Kan (1996) (see also Constantinides (1992), Frachot and Lesne (1993) and Frachot (1995)) and can be thought of as a multivariate version of the Cox, Ingersoll, and Ross (1985) model. More specifically, the linearity assumption implies that the state variables follow a multivariate square-root process under the risk-neutral probability. Special cases of this class are (among others) the Multifactor Gaussian model, the Cox, Ingersoll, and Ross (1985) model and also the Longstaff and Schwartz (1992) model. On the other hand, the Quadratic Gaussian model (El Karoui, Myneni, and Viswanathan (1992), Jamshidian (1993)) assumes that the yields are quadratic functions of some state variables where the dynamics of these state variables are Gaussian under the risk-neutral probability.

The main point is that, in all these models, the long zero-coupon rate is generally non stochastic in the sense that it can be expressed as a function of the underlying parameters but does not depend on the state variables.

Despite its simplicity, this remark has several consequences for both theoretical and empirical purposes. First, for empirical purposes, since actual data are likely to show stochastic long-term rates, any attempt to calibrate such traditional models will provide time-varying, stochastic parameters. As a matter of fact, the estimates capture this remaining uncertainty which *cannot* be taken into account by the model itself. At first sight, it may seem rather unimportant since, in industry practice, calibration is performed on a daily basis and parameter estimations are not expected to be stable over time. On the other hand, since the state variables are designed to capture the whole uncertainty of the model, what sense does it make to obtain stochastic parameters ? From an empirical point of view, perfect stability is certainly unrealistic but stability in itself is a desirable feature to assess the validity of the model.

Secondly, our goal was to provide a model where the long zero-coupon rate would be finite and would also be allowed to move up or down stochastically. In some sense, this was meant as an attempt to be in accordance with the observable behavior of the long-end part of the yield curve. Unfortunately, these attempts turned to be unsuccessful essentially because such a behavior (i.e. a finite long-term rate moving up or down stochastically) is *inconsistent* with the no-arbitrage assumption as shown by Dybvig, Ingersoll and Ross (1996).

Thirdly, the specific behavior of the long term rate has also some direct implications on the time-series properties of interest rate related data. In particular, the non-decreasing feature of the long term rate implies that these data should have no mean reversion and then reveal some unit root. This is consistent with all empirical studies.

We discuss the theoretical and practical consequences of the long rate properties for the existing models, especially those which are based on a state variable assumption. In this respect, we investigate the theoretical conditions under which a factor model would allow for a stochastic long zero-coupon rate. In all these models, the consequences of a stochastic long zero-coupon rate assumption are non trivial and also imply some important restrictions on the class of possible models. In particular, they require volatility functions very different from the traditional ones.

Moreover, we show that an appropriate correction of the Cox, Ingersoll, and Ross (1985) model provides some interesting features despite the puzzling behavior of the long-end part of the yield curve. In particular, it may help to find reliable explanations for some typical features observed in the calibration process and, more generally, in the econometrics of the existing models.

The remainder of the paper is organized as follows. Section 2 provides through a Heath, Jarrow, and Morton (1992) type framework the implications of the no-arbitrage assumption on the behavior of the long-end part of the yield curve. In section 3, we review in the light of these conditions some traditional models including those belonging to the Affine Yield and Quadratic Gaussian classes and we show how to correct them. Finally, section 4 discusses the empirical implications of our results.

2. SOME GENERAL RESULTS

We first start by a brief recall of the traditional no-arbitrage framework. Then some general results concerning the long-term rate are derived.

2.1. A No Arbitrage Framework. Following the seminal papers by Harrison and Kreps (1979) and Harrison and Pliska (1981), we characterize the absence of arbitrage by the existence of a risk-adjusted probability measure. Subsequently, when the randomness of the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$ where $\mathcal{F}_t$ is the information available at time t and P is the risk-adjusted probability measure, the dynamics of the price $B(t,T)$ of a zero-coupon bond maturing at time T are then given by the following stochastic differential equation (see for example Heath, Jarrow, and Morton (1992) and El Karoui and Geman (1991)):

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma(t,T)' dW_t, \quad t \leq T, \tag{1}$$

with the boundary condition $B(T,T) = 1$. $(W_t)_t$ is an n-dimensional P-standard Brownian motion adapted to $(\mathcal{F}_t)_t$; $\sigma(t,T)'$ is the transposed vector of an n-dimensional, continuous adapted process such that $\sigma(T,T)=0$, and whose derivative in the second argument (denoted by $\partial_2\sigma(t,T)$) is continuous and is an adapted process in t. Finally, r_t denotes the short rate defined as:

$$r_t = -\frac{\partial Ln\ B(t,T)}{\partial T}\ |_{T\searrow t}\ .$$

In a standard way, we assume that the initial yield curve (at time $t=0$) is given and we deduce from equation (1) the following expression for the zero-coupon prices:

$$B(t,T) = B(0,T)exp\left[\int_0^t (r_s - \frac{1}{2} \parallel \sigma(s,T) \parallel^2)ds + \int_0^t \sigma(s,T)'dW_s\right]. \tag{2}$$

Defining the spot forward rates by:

$$f(t,T) = -\frac{\partial Ln\ B(t,T)}{\partial T},$$

we easily derive from equation (2):

$$f(t,T) = f(0,T) + \int_0^t \partial_2\sigma(s,T)'\sigma(s,T)ds - \int_0^t \partial_2\sigma(s,T)'dW_s. \tag{3}$$

Put in terms of zero-coupon rates, this is also equivalent to:

$$Y(t,T) = Y^f(0,t,T) + \frac{1}{2}\int_0^t \frac{\parallel \sigma(s,T) \parallel^2 - \parallel \sigma(s,t) \parallel^2}{T-t}ds - \int_0^t \frac{[\sigma(s,T)-\sigma(s,t)]'}{T-t}dW_s, \tag{4}$$

where $Y^f(0,t,T)$ is the corresponding forward rate.

Equations (1), (2) and (4) are our fundamental equations. In particular, the short rate is obtained by taking $T \to t$ in equation (3) while the long-term yield corresponds to $T \to +\infty$.

2.2. The Long Term Rate. There are many ways to understand economically how a long term yield should behave. First, it can be viewed as a constant (possibly infinite) characteristic of the underlying economy. In this sense, it is natural for this long-term rate to be a deterministic function of some underlying parameters. As a matter of fact, we will show that most existing models implicitly assume a constant long-term rate. Unfortunately, a simple look at forward rate curves estimated for various countries is rather consistent with a finite-valued long-term rate which does actually vary over time in a stochastic way. In fact, it is clear that an infinite-maturity yield does not have a precise economic meaning but rather appears as a mathematical feature of the term structure; our purpose is to derive the financial consequences for the usual interest rate models of the fact that actual data are indeed inconsistent with a constant long-term rate. To summarize, we want to find out which constraints must be imposed on the volatility function $\sigma(s,T)$ in order to obtain a non deterministic long-term rate.

We recall that we are not interested here in the consol rate. Though the long-term (zero-coupon) rate and the consol rate are not independent from one another, they have different behaviors and must receive different mathematical treatments.

Let us denote the following limits (supposing that they are well-defined):

$$\begin{cases} \ell_t & = & \lim_{T\to+\infty} Y(t,T) \quad a.s \\ \\ \mu_\infty(t) & = & \lim_{T\to+\infty} \frac{1}{2}\frac{\|\sigma(t,T)\|^2}{T-t} \quad a.s \\ \\ \sigma_\infty(t) & = & \lim_{T\to+\infty} \frac{\sigma(t,T)}{T-t} \quad a.s \end{cases}$$

ℓ_t represents the long-term yield while $\mu_\infty(t)$ and $\sigma_\infty(t)$ are respectively its "drift" and "volatility" functions. Assuming sufficient regularity[1] for the volatility function $\sigma(t,T)$, we can write:

$$\ell_t = \ell_0 + \int_0^t \mu_\infty(s)ds + \int_0^t \sigma_\infty(s)dW_s.$$

Let us note that ℓ_t could be also defined as the limit of the zero-coupon yield $f(t,T)$ when T goes to infinity. These two limits are not equivalent from a strict mathematical point of view (see Dybvig, Ingersoll and Ross (1996)) but are equal when both of them exist. Thus we shall assume sufficient mathematical regularities to consider these two limits as equal.

We have the following lemma:

Lemma 1. *For any t, we have:*

$$\| \sigma_\infty(t;\omega) \| > 0 \Rightarrow \mu_\infty(t;\omega) = \infty$$

This is a straightforward consequence of the definitions of μ_∞ and σ_∞. If $\sigma_\infty(t)$ is strictly positive then $\sigma(t,T)$ goes to infinity with T. Hence $\mu_\infty(t)$ is infinite as well. As a result, a finite long-term yield has necessarily a zero volatility term.

The second lemma is less known although it was first established in the first version of Dybvig, Ingersoll and Ross (1996)'s paper (with a different proof).

Lemma 2. *For any t, we have:*

$$\mu_\infty(t) \geq 0 \ \ a.s$$

[1]We assume that there exists a L^2-integrable function $\phi(.)$ such as:

$$\left| \frac{\sigma(t,T)}{T-t} \right| < \phi(t).$$

The proof is straightforward since $\mu_\infty(t)$ is the limit of a sequence of positive terms. Consequently, we have the surprising result that the stochastic process which drives (ℓ_t) is necessarily given by:

$$\ell_t = \ell_0 + \int_0^t \mu_\infty(s)ds$$

with $\mu_\infty(s) \geq 0$ for all s. Let us note that this feature is totally *independent* of the underlying probability measure since the Brownian part has vanished. It is also clear that, if $\mu_\infty(.)$ is not infinite then $\sigma(t,T)$ behaves like $O(\sqrt{T-t})$ for infinite maturities.

A good way of summarizing these few lemmas is to derive the typical shape of the volatility and the yield curves:

	ℓ_t	Yield curve	Volatility curve
$\sigma_\infty(.) > 0$	infinite	$Y(t,T) \sim O(T-t)$	$\sigma(t,T) \sim O(T-t)$
$\sigma_\infty(.) = 0$ $\mu_\infty(.) = 0$	constant	$Y(t,T) \sim O(1)$	$\sigma(t,T) \sim O(1)$
$\sigma_\infty(.) = 0$ $\mu_\infty(.) > 0$ $\mu_\infty(.) < \infty$	non-decreasing	$Y(t,T) \sim O(\sqrt{T-t})$	$\sigma(t,T) \sim O(\sqrt{T-t})$

It is not obvious to determine which of these cases is the most reasonable one. Assuming a constant long-term yield is certainly as weird as taking it as a non-decreasing process. On the other hand, the first case provides a quite unrealistic shape for the yield curve. Surprisingly, almost all existing models have only considered the two first cases. Some specific examples are given by the Linear Gaussian Models, Cox, Ingersoll, and Ross (1985), Longstaff and Schwartz (1992) and Chen and Scott (1992) models and, as we shall show later, any one-factor models.

However, it would be natural to investigate the third case (i.e., a non zero μ_∞) especially because its features may be more realistic than those provided by the other two cases. In effect, the long term rate is not a time-independent constant (contrary to the second case) but tends to infinity in the long run while the volatility curve flattens for infinite maturity (contrary to the first case). To our knowledge, this intermediate case has never been systematically investigated and it is one of the goals of this paper.

The fundamental question concerns the financial interpretation of a non decreasing long-term rate. In order to provide some intuition, we can give a simple strategy which would be an arbitrage opportunity if the long term zero coupon rate could decrease. Let us invest one unit of numéraire (say 1$) in a zero coupon bond maturing at time T, corresponding to $1/B(t,T)$ units of this zero coupon bond. At time $t+1$, the value of this portfolio is then:

$$V_{t+1} = \frac{B(t+1,T)}{B(t,T)}.$$

When T goes to infinity, V_{t+1} converges towards the three following values depending on ℓ_{t+1} being below, equal or above ℓ_t :

$$V_{t+1}^{\infty} = \begin{cases} 0 & \text{if } \ell_t < \ell_{t+1} \\ \text{indeterminate} > 0 & \text{if } \ell_t = \ell_{t+1} \\ +\infty & \text{if } \ell_t > \ell_{t+1}. \end{cases}$$

Now, if the probability of $(\ell_t > \ell_{t+1})$ was strictly positive, then we could consider at time t the contingent claim whose terminal value at time $t+1$ is $Max(\ell_t - \ell_{t+1}, 0)$. This claim can be interpreted as a put option written on the long term zero coupon rate, maturing at time $t+1$ and whose exercise price is ℓ_t. Since the event $(\ell_t > \ell_{t+1})$ is assumed to have a non zero probability, then this option has a strictly positive price. If we sell this option to finance the previous portfolio, it is easy to see that we obtain a global portfolio which has zero value at time t and which provides (asymptotically) a positive cash-flow at time $t+1$ in all states. As a result, under the no arbitrage assumption, the event $(\ell_t > \ell_{t+1})$ must have zero probability.

However a non decreasing long term rate means that this rate is infinite in the long run. So we face the following alternative: the long-term rate is either constant over time or infinite. But, as it is clear from our previous table, there are two ways to go to infinity. The first one (i.e., the first row of our table) is much more ”violent” than the second one (i.e., the third row) and, in this last case, the corresponding yield and volatility curves have more realistic shapes. This is our basic motivation for focusing on this case.

The behavior of the long-term rate that is examined above is not related to any kind of state variable assumption and, even less so, to the short term rate being a state variable. It is exactly the goal of the next sections to investigate to what extent it is possible to build some specific models corresponding to this case that we now summarize as:

Main Assumption

- *The shape of the time-t volatility curve $T-t \to \sigma(t,T)$ behaves as $\sqrt{T-t}$ when $T-t$ grows to infinity. In particular, $\sigma_\infty(t)$ is equal to zero while $\mu_\infty(t)$ is strictly positive.*
- *Volatilities are stochastic*

The stochastic nature of $\sigma(t,T)$ allows for a stochastic behavior of the long end of the yield curve.

To do this, we now have to specify accordingly the volatility function. In past literature, the way these volatility functions have been specified can be divided into two distinct approaches. The first one does not assume any restrictions and leads to arbitrary volatility functions. An example is given by Amin and Morton (1994) through six different volatility forms which can be nested in the general class:

$$\partial_2\sigma(t,T) = [\sigma_0 + \sigma_1(T-t)]e^{-\lambda(T-t)}f(t,T)^\gamma.$$

By inspecting this formula, it appears that the asymptotic behavior implied by our main assumption is not consistent with any choice of the parameters $(\sigma_0, \sigma_1, \lambda, \gamma)$.

A less arbitrary but sufficiently flexible class of volatility functions is obtained through a state variable assumption as discussed in what follows. In this context, our main assumption will suffice to restrict considerably the class of admissible models.

2.3. A Simple Example. In order to clarify our discussion, we now detail the well-known model proposed by Cox, Ingersoll, and Ross (1985).

This model is governed by a single state variable identified to the short-rate (say) whose stochastic differential equation is:

$$dr_t = (\phi - \lambda r_t)dt + \sigma\sqrt{r_t}dW_t.$$

The volatility function has the form $\sigma(t,T) = A(T-t)\sigma\sqrt{r_t}$ where:

$$A(T-t) = \frac{1+k}{\nu}\frac{1-e^{-\nu(T-t)}}{1+ke^{-\nu(T-t)}}$$

with $\nu = \sqrt{\lambda^2 + 2\sigma^2}$ and $k = (\nu - \lambda)/(\nu + \lambda)$. λ is the mean-reverting parameter (under the risk-adjusted probability) and σ is a volatility parameter.

Our main assumption is clearly not satisfied since μ_∞ equals 0. Although the empirical implications of our conditions are discussed later in the paper and in order to motivate our subsequent derivations, we can give a brief account of what may happen when a CIR model is estimated in a context of time-varying long-term yields. Taking ν equal to 0 is actually the only way to prevent the volatilities from converging, meaning that the long end part of the yield curve is "sufficiently stochastic". Unfortunately, it is theoretically inconsistent since it implies that both the mean-reverting and volatility parameters of the short rate are equal to 0 (the ultimate consequence is that the short rate is deterministic). Since the same observation can be made for the Vasicek (1977) model or, more generally, the Linear Gaussian model, it might explain why mean-reverting parameters are often found to be close to 0. It just reflects that the long-term rate is not constant over time.

3. MULTIVARIATE FACTOR MODELS

In this section, we consider the case where state variables are assumed to govern the dynamics of the term structure in the light of our previous discussion on long-term rate. We show that these dynamics should respect some constraints. The class of affine factor models introduced by Duffie and Kan (1996) (see also Frachot and Lesne (1993)) is fully detailed.

Our challenge is to find a model where the long term rate is not a constant. Indeed, we believe that it is the crucial point in order to reconcile thoretical models and empirical issues.

We shall see that the joint hypothesis of no-arbitrage opportunities and non constant long term rate imposes some strong constraints on the models one can derive. In the next section, we shall show that these constraints are at the root of many empirical issues.

3.1. Long-Term Rate and State Variables. The first question to address is the number of state variables we should put in a plausible model. One-factor models are widely used in finance industry because they are generally simple to implement numerically. However when we use a one-factor model, we restrict the way the long end part of the term structure may behave:

Lemma 3. *In a one-factor model, the long-term rate (if it exists) cannot be stochastic.*

Let us note X_t the state variable. If ℓ_t was stochastic, then ℓ_t should be some function of X_t but, since its instantaneous volatility is equal to 0, by Ito's lemma $\frac{\partial \ell}{\partial X}$ would be equal to 0. As a result, ℓ_t is deterministic and then there is hardly any chance to find a one-factor model satisfying our main assumption relative to the long-term rate. We thus conclude that at least two factors are necessary but even in this case we always have to impose that one of the state variables has a zero volatility:

Lemma 4. *In a multi-factor model, the long-term rate (if it exists) cannot be stochastic if the volatility matrix of the set of state variables is non singular.*

This lemma is also easy to prove. Since the volatility of the long-term rate is zero, it cannot be a non singular function of the state variables unless there exists a singularity in the volatility matrix of our state variables.

Keeping these lemmas in mind, we can now detail how to build a multi-factor model which would respect our main assumption.

3.2. Multivariate Affine Factor Models. In this section, we concentrate on the most tractable among the affine factor models proposed in past literature. These models are affine in the sense that the yield of any zero-coupon bond is assumed to be an affine function of a set of state variables.

This class has many advantages. First it is sufficently large to contain some traditional models such as Vasicek (1977), Cox, Ingersoll, and Ross (1985), Duffie and Kan (1996) and to encompass the class of Linear Gaussian Models proposed by Jamshidian (1989) or El Karoui and Lacoste (1992). Secondly, because of linearity, these models lead to tractable calculations. Thirdly, from an empirical point of view, this class is consistent with the results obtained from a "factor analysis" of the term structure as initiated by Litterman and Scheinkman (1991). Indeed, the empirical covariance matrix of a set of yields typically shows two or three significant eigenvalues while the others are numerically very close to 0. This exactly means that there exist some affine relationships between these yields and this provides a strong case for the affine factor models (see Frachot, Janci, and Lacoste (1993)).

Notation. The affine factor structure will be written as:

$$B(t,T) = exp - [A(T-t)' F_t + b(T-t)] \tag{5}$$

where (F_t) denotes a two-dimensional process defined by:

$$dF_t = D(F_t)dt + V(F_t).dW_t, \tag{6}$$

where $D(F_t)$ is a $n \times 1$ drift function and $V(F_t)$ is a $n \times n$ matrix of volatility ($n = 2$ or 3).

As shown in Duffie and Kan (1996) or Frachot and Lesne (1993), combining equations (5) and (6) with equation (1) yields the following two results. First, the $n \times n$ matrix $V(F_t).V(F_t)'$ is affine in F_t as well as the drift function $D(F_t)$. We denote $V(F_t).V(F_t)' = (\alpha'_{ij}.F_t + \beta_{ij})_{0 \le i,j \le n}$ and $D(F_t) = \phi - \Gamma.F_t$.

Secondly, the function $A(.)$ is solution of an ordinary differential equation which can be written as:

$$\partial A(x) = \partial A(0) - \Gamma' A(x) - \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij} A_i(x) A_j(x). \tag{7}$$

The intercept $b(T-t)$ is also solution of an ordinary differential equation:

$$\partial b(x) = -\frac{1}{2} \sum_{i,j=1}^{n} \beta_{ij} A_i(x) A_j(x) + \phi' A(x). \tag{8}$$

These equations define the whole model which is thus parametrized by ϕ, Γ, α_{ij}, and β_{ij}. For the moment, we are only interested in time-homogeneous models and subsequently our parameters are not supposed to be time-varying. This point will be discussed later in the paper.

Long-Term rate And Two-Factor Affine Models. Here n is equal to 2. We first discuss to what extent the long-term rate could be taken as one of the state variables. Since the short rate r_t is also a natural choice, we perform a basic change of variables to rewrite the factor structure of equation (5) as:

$$-Ln\ B(t,T) = A_1(T-t) r_t + A_2(T-t) \ell_t + b(T-t), \tag{9}$$

with the obvious boundary conditions:

$$\begin{cases} A(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad b(0) = 0 \\ \\ \partial A(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \partial b(0) = 0 \\ \\ \lim_{x \to \infty} \partial A(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \lim_{x \to \infty} \partial b(x) = 0 \end{cases} \tag{10}$$

Since we observed that the instantaneous volatility of the long-term rate is equal to 0, we have $\alpha_{22} = \alpha_{12} = 0$. The dynamic part (6) is thus rewritten as:

$$d\begin{pmatrix} r_t \\ \ell_t \end{pmatrix} = \left[\phi - \Gamma.\begin{pmatrix} r_t \\ \ell_t \end{pmatrix}\right]dt + \begin{pmatrix} \sqrt{\alpha_{11}'\begin{pmatrix} r_t \\ \ell_t \end{pmatrix} + \beta_{11}} & 0 \\ 0 & 0 \end{pmatrix}.dW_t. \tag{11}$$

It is now quite simple to see that our parameters (especially Γ and α_{11}) can not be given arbitrary. These constraints come from the type of ordinary differential equation satisfied by the coefficients $A(x)$. This is actually a Riccati-type equation which appears to be consistent with our boundary conditions (10) only in a very few cases.

Rewriting this equation Riccati equation yields:

$$\partial A(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \gamma_1 A_1(x) - \gamma_2 A_2(x) - \frac{1}{2}\alpha_{11}A_1(x)^2, \tag{12}$$

where we denote $\Gamma' = (\gamma_1, \gamma_2)$ with γ_1 and γ_2 two 2×1 vectors. Intuitively, the three vectors γ_1, γ_2 and α_{11} cannot be linearly independent because functions $A_1(.)$ and $A_2(.)$ do not have the same asymptotical behavior. Since the volatility function $\sigma(t,T)$ behaves like $O(\sqrt{T-t})$, $A_1(x)$ is proportional to $\sqrt{x}$ when x grows to infinity while $A_2(x)$ is proportional to x since its derivative converges to 1. So in the right-hand side of the previous equation, the x and $\sqrt{x}$ terms must drop.

Lemma 5. *Vectors γ_1, γ_2, and α_{11} belong to the same uni-dimensional subspace generated by* $(1,-1)'$.

The proof is straightforward. Lets first remark that we want γ_2 to be different from 0; if not, from equation (11), the drift of ℓ_t would be zero and ℓ_t itself would be deterministic. Secondly, if γ_1 and α_{11} were not proportional to γ_2, we could project equation (12) on the subspace orthogonal to γ_2, and, since $\partial A(x)$ converges to a constant vector $(0,1)'$, then $A_1(x)$ and therefore $A_2(x)$ would have limits as x goes to infinity. This condradicts the boundary condition (10) where it is expected that $A_2(x)$ diverges. Finally, taking $x \to \infty$ in equation (12), we see that γ_1, γ_2, and α_{11} are proportional to $(1,-1)'$.

As a result, the Riccati equation is much simpler, namely:

$$\partial A(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}\left[\frac{1}{2}\alpha A_1(x)^2 + \lambda_1 A_1(x) + \lambda_2 A_2(x)\right]. \tag{13}$$

Let us notice that $\frac{1}{2}\alpha A_1(x)^2 + \lambda_1 A_1(x) + \lambda_2 A_2(x)$ should converge to 1 when x tends to infinity. This implies that the remaining intercept $b(T-t)$ is actually equal to zero. Indeed, $b(T-t)$ satisfies $\partial b(x) = -1/2\beta_{11}A_1(x)^2 + \phi'.A(x)$ and should converge to 0 in order to be consistent with our boundary conditions. Since $A_1(x)$ and $A_2(x)$ both diverge, the only way for $\partial b(x)$ to converge is to be proportional to $\frac{1}{2}\alpha A_1(x)^2 + \lambda_1 A_1(x) + \lambda_2 A_2(x)$, and since its limit must be zero, it must be identically equal to 0. So $\phi = \beta_{11} = 0$.

Let us summarize our results. First, our assumption that the long-term rate is a state variable appears to be quite powerful since it severely restricts the size of our set of parameters (there remain three parameters α, λ_1, and λ_2).

Regarding the coefficients $A_1(x)$ and $A_2(x)$, we see that they both diverge to infinity, in total contradiction with what was obtained in the CIR case.

From equation (13), functions A_1 and A_2 are constrained to satisfy $\partial A_1(x)+\partial A_2(x) = 1$, or $A_1(x) + A_2(x) = x$. It enables us to reduce equation (12) to a single unidimensional differential equation:

$$\partial A_1(x) = 1 - \lambda_1 A(x) - \frac{1}{2}\alpha A_1(x)^2 - \lambda_2(x - A_1(x)), \tag{14}$$

which is very close to the differential equation obtained in the CIR case except that an additional parameter λ_2 appears and is expected to capture the asymptotic behavior of the long-term rate. Let us also note that this equation is still a Riccati equation but, to our knowledge, it has no closed-form solution. This also enables us to write the factor structure for the yields in the following way:

$$Y(t,T) = \ell_t + \frac{A_1(T-t)}{T-t}(r_t - \ell_t).$$

This sounds exactly like the standard conclusions obtained from the empirical factor analysis of the term structure. Indeed, there appears a "level factor" ℓ_t whose corresponding loading is constant across maturities and, second, a "slope factor" $r_t - \ell_t$ whose corresponding coefficient decreases from 1 (for $T - t = 0$) to 0 (for $T - t = \infty$). This is nicely consistent with the empirical results first found by Litterman and Scheinkman (1991) (and many others after them).

Unfortunately, although the factor structure presents these appealing features, the dynamic part of the model is rather inconsistent. As a matter of fact, the stochastic diffusion equations are of the form:

$$\left\{\begin{array}{lcl} dr_t & = & -\lambda_1(r_t - \ell_t)dt + \sqrt{\alpha(r_t - \ell_t)}dW_{1t} \\ \\ d\ell_t & = & -\lambda_2(r_t - \ell_t)dt \end{array}\right. \tag{15}$$

From equation (14), we observe that parameters α and λ_2 should have opposite signs because the $O(x)$ part must be eliminated away. If we suppose $\alpha > 0$, then $r_t - \ell_t$ is always positive and λ_2 is negative. However, this is clearly a puzzling process since 0 is an absorbing bound for $r_t - \ell_t$. In particular, the technical conditions ensuring that the spread $r_t - \ell_t$ never hits 0 are not satisfied. As a conclusion, the long-term rate *can not* be taken as one of the state variable or, said differently, the long-term rate is necessarily infinite.

3.3. A Modified Cox, Ingersoll and Ross Model. The question is how to improve this model while preserving its good properties (mainly its factor structure). Let us consider the following bi-variate dynamics:

$$\left\{\begin{array}{lcl} dX_t & = & (\phi - \lambda X_t)dt + \sigma\sqrt{X_t}dW_t \\ dZ_t & = & (-\tilde{\lambda} X_t)dt \end{array}\right. \tag{16}$$

It is formally equivalent to the previous case if $X_t = r_t - \ell_t$ and $Z_t = \ell_t$ except that we want ϕ to be different from zero. In particular, this remaining parameter might be taken time-dependent in order to match the initial yield curve. Under this parametrization, the model can be understood as a modified version of the CIR model where the CIR case is obtained for $\tilde{\lambda} = 0$. It has also some similarities with the Ritchken and Sankarasubramanian (1995) model since Z_t is more or less equal to the sum of the square of the volatility of (X_t). The corresponding factor structure is preserved; indeed, equation (16) implies that:

$$-Ln\ B(t,T) = A(T-t)X_t + (T-t)Z_t + b(T-t), \tag{17}$$

with

$$\begin{array}{rcl} \partial A(x) & = & 1 - \lambda A(x) - \frac{1}{2}\sigma^2 A(x)^2 - \tilde{\lambda}x \\ \partial b(x) & = & \partial b(0) + \phi A(x) \end{array} .$$

Now the long-term rate is infinite since ϕ is different from zero[2] while the yield and forward curves have parabolic shapes for large maturities. Moreover, this model departs from the CIR case by only one parameter $\tilde{\lambda}$. However, it is still unsatisfactory to see that the second state variable Z_t is always a non-decreasing process. As X_t and Y_t govern the whole term structure of interest rates, they might drive it to infinity !

At this point, the question which arises from our previous derivations is the following: is it possible to find a consistent model where the long-term rate is not deterministic and where the term structure of interest rates does not go to infinity ?

There are many ways to answer this question. First one can argue that more than two factors should be necessary. Unfortunately, our attempts in this direction have been unsuccessful since we always obtain that one of the state variables has zero volatility and positive drift. As it enters linearly in $Y(t,T)$, the whole term structure likely goes to infinity. Whether *any* affine factor model has this problem is left as an open question. Alternatively, we can try to use non linear models such as Quadratic gaussian models but our attempts were unsuccessful as well (the mathematical reasoning is roughly the same). Let us also remark that, even in non linear models we may encounter some strange results. Using the Jensen inequality in the expression:

$$B(t,T) = E_t \left(\exp - \int_t^T r_s ds \right)$$

we obtain:

$$Y(t,T) < \frac{1}{T-t} \int_t^T E_t r_s ds.$$

Taking the limit when T goes to infinity, we see that $E_t r_s$ is unbounded as s goes to infinity when the long-term rate is infinite.

[2]For ease of exposition, we have assumed all along that ϕ is constant. It can obviously be taken as time-dependent to match the initial yield curve. In this case, the intercept $b(t,T)$ depends on t and T separately.

4. EMPIRICAL IMPLICATIONS

This section is devoted to a reexamination of some various empirical points in the light of the previous discussion. Here we wish to develop the idea that numerous issues encountered in empirical work can be reliably related to the behavior of the long-term rate, and secondly that our modified CIR model despite its unsatisfactory dynamics is likely to be better specified than the traditional CIR model. Starting from our modified version of CIR, we review a large variety of problems in a rapid manner even though some of them would might deserve a specific paper.

4.1. A Discrete Framework. In past literature, two ways of testing models of the term structure of interest rates have emerged along depending on whether both cross-section and time-series consequences of the underlying model are used jointly or not. Brown and Dybvig (1986) and Dahlquist and Svensson (1993) use the cross-section restrictions imposed by the CIR model and, thus estimate the underlying parameters by matching theoretical prices (of a set of coupon bonds) to their actual values (by use of a non-linear least squares procedure). Longstaff and Schwartz (1992) test also some cross-sectional restrictions using the Generalized Method of Moments (GMM). In contrast, other work including those of Heston (1989) exploits the additional information provided by the state variable process while Pearson and Sun (1990) and Chen and Scott (1993) nest this approach within a general multifactor estimation scheme.

As it is well-known, the cross-section part involves risk-neutral parameters while the time-series part includes the additional "market-price of risk". At first sight, the dynamic processes derived in section 3 do not provide any information on the true processes driving the yields since they are given under the risk-neutral probability which is irrelevant for time-series issues. To come back to the historical probability measure, one has to specify more precisely the underlying economy. This can be done by connecting the arbitrage model to an equilibrium framework where the "market price of risk" could be derived. We assume that the change of probability measure does not alter the structure of the dynamic process (i.e. only mean-reversion parameters are modified) and thus we interpret directly the equations given in section 3. In particular, when we allude to mean-reversion parameters, we refer to their risk-neutral versions.

In order to discuss the empirical issues, one must derive a discrete version of the model (16) described in the previous section. As a matter of fact, it is well known that the conditional expectation and the conditional variance of the model (16) are linear in terms of the state variables. So the discretized version can be written:

$$\begin{cases} X_t = \Phi_1 + \rho_1 X_{t-1} + \sigma_1 \sqrt{X_{t-1} + \beta_1} \epsilon_{1t} \\ Z_t = \Phi_2 + \rho_2 X_{t-1} + Z_{t-1} + \sigma_2 \sqrt{X_{t-1} + \beta_2} \epsilon_{2t} \\ -Ln\ B(t,T) = A(T-t) X_t + (T-t) Z_t + b(T-t) \end{cases} \tag{18}$$

where all parameters are some obvious function of the continuous-time paramaters ϕ, λ, $\widetilde{\lambda}$,

σ and ϵ_{1t} and ϵ_{2t} are two (correlated[3]) white noises satisfying:

$$\begin{aligned} E_{t-1}\epsilon_{1t} &= 0 \\ E_{t-1}\epsilon_{2t} &= 0 \\ V_{t-1}\epsilon_{1t} &= 1 \\ V_{t-1}\epsilon_{2t} &= 1 \end{aligned}$$

4.2. Low Mean Reversion Issue. Some of the issues encountered in empirical work concern the low (or even negative) mean-reversion and the inability of some traditional models to match with long maturities. Low or negative mean-reversion from cross-section data (observed, for example, in Dahlquist and Svensson (1993) or Gouriéroux and Scaillet (1994)) can be explained from the remarks made in section 2: in our framework, the factor loadings (i.e. the sensitivity of bond prices to the state variables) should diverge when maturity $T-t$ tends to infinity while in usual models they behave as $O(1)$. In some sense, a non positive mean-reversion parameter is, in most cases, the only way to prevent the factor loadings from being convergent, but a zero mean-reversion parameter implies generally that the forward curve behaves as $O(T-t)$ and not as $O(\sqrt{T-t})$. Consequently, misspecification occurs systematically and it helps explaining why these usual models do not fit properly the long end of the yield curve (as noted by Longstaff and Schwartz (1992) and Chen and Scott (1993) for the CIR-like models). Conversely, when one uses shorter maturities (e.g. maturities under 10 years like in El Karoui, Geman and Lacoste (1995)), mean reversion parameters have greater values and this misspecification is less crucial. It then proves that low mean reversion maturities are strongly connected with the long term yield.

These results are confirmed when additional information is used by taking into account the state variable process. As illustrated by Pearson and Sun (1990) or Chen and Scott (1993), the estimation of multi-factor versions of the CIR model implies low mean-reversion for one of the state variables. In short, as noted by Chen and Scott (1993), a zero mean-reversion in one of the factors is the only way to explain the observed variation of long-term interest rates. These results are perfectly consistent with our model (18) since the second state variable Z_t has no mean reversion.

On the other hand, taking a zero mean-reversion in traditional models leads to an unrealistic behavior for both the forward and spot yield curves. In contrast, our model reconciles both zero mean-reversion and asymptotically flat yield curves.

4.3. Factor Analysis of the Yield Curve. As another example of application, let us come back to the empirical factor analysis of the term structure pioneered by Litterman and Scheinkman (1991): it consists in computing the eigenvalues and eigenvectors of the empirical covariance matrix of a set of yields of given maturities (see Frachot, Janci, and Lacoste (1993)). Our model (18) gives a consistent factor structure where the two factors are X_t and Z_t. In terms of yields,

[3]This correlation depends on the state variables. More specifically, the conditional covariance matrix of $(X_t, Z_t)'$ is affine in X_{t-1}. As a result, the correlation is a rather complicated function of X_{t-1}.

this factor structure is given by:

$$Y(t,T) = \frac{A(T-t)}{T-t} X_t + Z_t + \frac{b(T-t)}{T-t}$$

It predicts that the factor loading associated to the state variable (Z_t) should be constantly equal to 1 for any yield. So it is not surprising that one of the eigenvectors thus obtained should be proportional to $(1, ..., 1)'$. What is less straightforward is to understand why this is in all cases the *first* eigenvector (i.e., the eigenvector associated to the strongest eigenvalue). As a matter of fact, it suffices to understand that a factor analysis upon a set of yields which are a mixture of explosive and non explosive components will overweigh the explosive part. Since (Z_t) has no mean-reversion, it is actually explosive and, thus plays the predominant role in any factor analysis. Let us also note that, though the instantaneous volatility of (Z_t) is equal to 0, its unconditional variance is unbounded as time goes to infinity and dominates the variance of the other (non-explosive) state variable X_t. Standard conclusions from a factor analysis of the yield curve for hedging strategies should deserve further comments. Indeed, for hedging purposes, we are more interested in the instantaneous volatility and thus a portfolio has to be hedged against (X_t) while, at the same time, when we refer to a factor analysis of the yield curve, the level factor (i.e., (Z_t)) is considered as the most important factor to be hedged against.

4.4. Parameter Instability. As noted by Pearson and Sun (1990), Dahlquist and Svensson (1993) or Gouriéroux and Scaillet (1994), estimates show great instability over time. It is generally interpreted as an indicator of misspecification and can be obviously related to the necessity for the parameters to capture the remaining uncertainty due (among other explanations) to the stochastic long-term rates. In particular, the estimates share a common component (i.e. (Z_t)) which may explain why they appear so highly correlated to one another (as pointed out by Longstaff and Schwartz (1992)) and why they are sometimes so difficult to recover with good precision.[4] As an example, Dahlquist and Svensson (1993) conclude that the parameters of the Longstaff and Schwartz (1992) model are difficult to identify empirically.

However, though it has long been recognized that one of the drawbacks of the CIR model (say) was due to its constant long-term rate (see, for example, Brown and Schaefer (1994)), economists have never attempted to progress towards a better specification. It may be explained by the fact that, in industry practice, calibration is performed daily from cross-section data (by use of observed prices) rather than from a time-series approach (by use of historical data). In doing so, estimates are not expected to remain (and actually are not) constant. As noticed in previous sections, even for industry practice, daily calibration does not suffice to alleviate this misspecification since the shape of the volatility function will remain rather different from what it should look like. Moreover, it likely explains why the calibration process

[4] It is illustrated by the findings of Brown and Schaefer (1994): they fit a CIR model to the term structure of real interest rates (using British governement index-linked securities) and show that estimates of the mean-reversion parameter are highly correlated with the short rate.

itself (within cross-section data) raises implementation difficulties as some parameters may not be recovered easily nor with good precision (especially mean-reversion parameters).

4.5. Time Series Issues and Unit Root Problems. Let us finish by addresing some issues raised by economists with respect to the time-series part of the various models. In this kind of approach, economists are motivated by the consideration of dynamical process followed by interest rates and the way of making reliable predictions of their future paths. They generally do not refer to the type of financial models discussed here and, subsequently, formulate some ad-hoc dynamical processes. There exists of course a wide literature on interest rate dynamics and we do not try to cover it all, but it is interesting to derive the consequences of our model for this purpose.

Our discretized model (18) appears as a two-dimensional vectorial autoregressive model of order 1, that is a model where conditional expectations depend linearly on the current values. The process (X_t) is stationary (i.e., non explosive) as long as the (discrete) mean-reversion parameter ρ_1 is not equal to 1 while the second state variable (Z_t) is necessarily non stationary (i.e., explosive) since it has no mean-reversion. As a result, any yield can be decomposed in one non-stationary and one stationary components. This is consistent with the "unit-root literature" where such a feature has been exhibited for a long time (see for example Campbell and Shiller (1991)). Importantly, this unit-root feature is independent of the probability measure and should be preserved under the historical probability measure.

Moreover, such a framework is able to predict that, for example, any spread of two different yields is likely to be stationary and this is in accordance with most empirical works.

In the same way, the volatility of the short rate should be stationary as well. More generally, our model suggests that an n-dimensional autoregressive model should reveal at least one unit-root and symetrically $n-1$ cointegrated relations. Let us also note that the variance matrix of the residuals should be close to singularity (since only two state variables are used); as the procedure traditionally used to estimate these econometric models (Johansen (1988)) requires non-singularity, this problem could plague the stability of the estimates.

As a final remark, we can also reinterpret the fact that volatilities are found time-varying. The whole AutoRegressive Conditional Heteroskedasticity (ARCH) literature is based on the empirical evidence that volatilities cannot be considered as constant over time and are closely linked to yields. As a matter of fact, we provide a simple explanation of this empirical evidence: as mentioned in previous sections, stochastic volatilities are one of the necessary conditions to ensure that the long-end part of the yield curve does not remain constant over time. As a result, the presence of strong ARCH effects is natural in a context of interest-rate related data (see Frachot (1996)). Furthermore, our model (18) gives the functional form that should be retained to model the stochastic feature of the volatilities: they can always be expressed as linear combinations of some particular yields.

5. CONCLUDING REMARKS

Traditional models of the term structure of interest rates generally assume that the long-term zero-coupon rate is constant or deterministic. If we want to depart from this unrealistic case, then the no-arbitrage assumption implies some intriguing results. In particular, the long-term rate is necessarily a non-decreasing process. By investigating this case through the Affine Yield models, we derive what the typical features of these models must be. In particular, we develop a modified version of the Cox, Ingersoll, and Ross (1985) model. Though this new model raises other issues, it sheds some light on the econometric problems encountered in practice when models of the term structure of interest rates are estimated. The question of whether the Affine Yield class as well as the Quadratic Gaussian Models are rich enough to provide a fully-consistent model is left as an open problem.

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